Building Conceptual Understanding in Precalculus
by Mairead Greene and Paula Shorter

Background

Picking up a random student transcript these days, you will often see that the student has completed a math course by the name of Precalculus. However, on the vast majority of those same transcripts (85-90% at Rockhurst), you will not see a Calculus course listed anywhere. It naturally compels us to ask the question, "Why are students taking a Precalculus course if they are not continuing on to Calculus?". Although the "Pre-calculus" course was originally developed to prepare college students for Calculus, the truth is that today most Precalculus students nationwide do not continue on to complete a calculus course. (Even of those college students nationwide who \textit{intend} on completing both precalculus and calculus, only 30-40% successfully complete the two course sequence.) At the same time, the number of students taking Precalculus has increased and diversified - both at Rockhurst and nationwide. So, who is taking precalculus and why are they taking it?

Currently, about 65% of Rockhurst students satisfy their math proficiency requirement (one college level math course required by our core) by taking Precalculus, and these students range from business majors to nursing majors with any number of Biology majors, Theology majors, Psychology majors, and a variety of others in between. In fact, several of these disciplines at Rockhurst (business and nursing, for instance), specifically require that their students satisfy the math proficiency requirement by taking Precalculus.

If these programs are not requiring or encouraging their students to enroll in this course to prepare for Calculus, what do they see in Precalculus? One of the major reasons for the popularity of this course among these diverse disciplines at Rockhurst is the work that our math department has done over the years in building a vision for the Precalculus course that supports the aims and objectives of other disciplines while still developing the desired mathematical thinking of students in what is most likely their final college math course. No single subset of quantitative skills or methods would be applicable to a large number of disciplines at once, nor would such learning objectives serve any student for very long past the final exam in Precalculus. Our vision, instead, centers around two main beliefs: 1) the most relevant, useful and lasting outcome that students will take away from this course is achieving an understanding of the meaning of mathematical concepts (i.e., \textit{conceptual understanding}), and 2) the concepts of this course should build and support the understanding of mathematical modeling of data.

A student who achieves conceptual understanding in our course is understanding the meaning of a concept well enough (or deeply enough) to be able to "adapt, modify and expand"\footnote{One characteristic of understanding concepts is the ability to adapt, modify and expand on a concept according to "Analysis of the Learner Characteristics of Students Implied by the Perry Scheme" by Cornfeld, J.L. and Knefelkamp, L.L. Copyright (c) 1979 by L. Lee Knefelkamp. According to Perry and Knefelkamp this level of understanding enables a student "to investigate and compare things and to make judgements about adequacy or inadequacy, appropriateness or inappropriateness."} that concept in order to apply it in novel situations or novel ways (within an application or within a purely mathematical context). For the study of mathematics to be more relevant and applicable for non-mathematics students this is essential. It is also extremely important for future mathematics majors. It is impossible to ever expose a student to all of the situations in which they may need to apply a given concept in their discipline (even if that discipline is mathematics). For this reason, it is important that a student is truly
understanding each mathematical concept and is able to reason from that understanding to answer questions - as opposed to simply memorizing (or mimicking) a few methods utilizing that concept.

In society today, we have the ability to collect, store and access more data than was even imaginable fifty years ago. Students will encounter data in almost any discipline they choose/have chosen to study. It is imperative that they are able to use and interpret data effectively while also critically analyzing the conclusions that others in their discipline have drawn from data. Mathematical modeling of data is a way to approach a quantitative question by examining appropriate observed data and drawing reasonable conclusions based upon mathematical analysis of that data. This mathematical analysis involves examining - both numerically and graphically - observed data that relates two varying numerical quantities (for example, years since 1970 and minimum ozone readings - as shown in boxed example below), recognizing any relationships that may exist between these two varying quantities (for example, as years since 1970 increases at a steady rate minimum ozone readings decrease by less and less), finding a mathematical function (specifically, a mathematical, symbolic description of how to relate two changing quantities) that describes that specific relationship, using that mathematical function to draw conclusions or make predictions where we do not have observed data (for example predicting the minimum ozone readings in the year 2020) and finally evaluating the reasonableness of those conclusions/predictions based upon the accuracy of the model. By it's very nature, mathematical modeling requires students to use conceptual understanding to think critically about a wide variety of quantitative questions as the questions that we ask and the data available to answer them are extensive, diverse and ever changing.

**Modeling Ozone Data**


Measurements are taken from approximately Mid Sept to Mid Oct in Antartica.

**Use modeling function to make predictions:**

- Predicted minimum ozone reading in year 2020:
  
  \[ f(50) = 593.15(50)^{-0.533} = 73.7 \text{ DU} \]

- Predicted year when minimum ozone reading reaches a level of 50 DU:
  
  \[ f(t) = 593.15t^{0.533} = 50 \]
  
  \[ t = 103.6 \text{ years since 1970} = 2073.6 \]

This chapter will focus on the challenge of providing the learning environment and student activities that help our students achieve conceptual understanding in Precalculus. We will describe why we believe that conceptual understanding can be built through an active-learning, student centered Precalculus classroom where students are challenged to discover concepts and connect their discoveries with their own prior understanding and the world around them. We will also share our approach to assessing whether our students have achieved conceptual understanding and the results of our assessment efforts to date.
Building Conceptual Understanding

Most students enter our Precalculus class with the belief that "the method" is everything. It is always surprising to us when some students glance at a question and almost immediately pronounce that they "don't know how to do it". We know those students don't actually mean this because they couldn't possibly know this without spending more than ten seconds thinking about the question. What those students mean is that they do not immediately know "the method". They are attempting to recognize and classify the question as a certain "type of problem" and then access the memorized sequence of steps ("the method") necessary to calculate the answer. If the question is not of a type that they recognize, then they conclude that they are unable to answer the question. This is worrying not only because those students do not understand how to read a question, think about it, discuss it and attempt to develop an approach to the question themselves building from the concepts involved; but more importantly because they don't even consider that process a possibility!

Our general sense that many students were coming to our course approaching problems in this very restricted way was confirmed when we reviewed the results of an online attitude survey that our students took during the first few days of the semester. One question asked students to: "Indicate how much you agree or disagree: In order to solve a challenging math problem, I NEED...:

a. To carefully analyze different possible solutions.
b. To have lots of practice in solving similar problems.
c. To understand other students' mathematical thinking.
d. To have natural talent for mathematics.
e. To try multiple approaches to constructing a solution.
f. To remember a lot of examples that I might use in constructing a solution.
g. To use rigorous reasoning.
h. To have freedom to do the problem in my own way.
i. To work hard"

Students chose an integer value on a scale of one through seven for each of the options (a) - (i). The response "1" corresponded to "Not at all" and the response "7" corresponded to "Very Much". Over all seven sections of our Precalculus course (during the Fall 2009 semester) the highest scoring option was (i) "To work hard" with an average response of 6.316 on the scale of 1 to 7. The next highest averages were options (b) "To have lots of practice in solving similar problems" and (f) "To remember a lot of examples that I might use in constructing a solution" with overall average responses of 6.007 and 5.549, respectively. These two options - both of which indicate a perceived need to recognize and categorize a problem in order to solve it - were ranked significantly higher than the remaining options, which together averaged a response of only 4.5 over all sections. A different question on this online attitude survey supported our belief that most students rely heavily on memorizing and using methods to solve problems. On this question students scored "Learning specific procedures for solving math problems" and "Memorizing the sets of facts important for doing math" as "Very Important" with average responses of 5.43 and 5.50, respectively, on a scale of 1="Not at all Important" to 7 = "Extremely Important".

Upon reflection, this should not be surprising to us. In a typical, traditional mathematics classroom, the instructor begins by presenting definitions and explanations of new concepts to the class, perhaps also providing some level of rationalization or context by linking the new concepts to past and future classwork. The instructor then works some sample problems for the students, illustrating ways in which
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this new concept can be applied to answer questions or solve problems. During this process, students may be active in some way, but generally in this traditional classroom, it is the instructor alone that actively engages in the challenging task of reasoning from his/her own understanding of a concept to the solution of a problem or the development of a new concept. The students, on the other hand, are passively watching most of this process. They may understand the teacher's explanations and reasoning. They may even agree that the steps the teacher took build logically from the concept. They may even by actively working through portions of the problem (e.g., completing algebraic work, computations, etc.), but they never have to reason themselves from their own understanding of the concept to arrive at an approach and an answer to a given question. When these students are then faced with a question or problem to solve themselves, they are necessarily approaching it by considering the sample problems that they've seen, selecting one that is most like this new question/problem and using (perhaps even adapting slightly) "the method" of that sample problem to find the answer. In doing this, they are essentially mimicking their teacher’s reasoning which requires only minimal understanding of the concepts involved. Students come away from this type of learning able to solve a lot of useful problems, but their abilities are limited to the types of problems that they've already seen. Another concern, especially in a discipline that is widely accepted as building strong problem solving and critical thinking skills, is whether this type of learning truly requires critical thinking on the part of the students.

The effects of our students' previous mathematics learning experiences could again be seen when reviewing the results of the online attitude survey administered at the beginning of the semester. Students "strongly agreed" that they "learn mathematics best" when "The instructor explains the solutions to problems," and when "The homework assignments are similar to the examples considered in class." These two options scored an average response over all sections of precalculus in Fall 2009 of 6.373 and 6.379, respectively, on a scale of 1 to 7 (where "1" signified "Strongly Disagree" and "7" signified "Strongly Agree").

In our precalculus classroom, we strive to push our students beyond this traditional approach to learning mathematics so as to broaden their problem solving abilities and to strengthen their critical thinking skills. Specifically, we want them to be able to reason independently from a deep understanding of concepts (or prior knowledge) to answer questions and solve novel problems. Providing plenty of opportunities (some guided and some unguided) for students to engage in this type of reasoning in novel situations (where there is no example to work from) slowly builds the confidence and ability to reason in this way independently. It is this very experience, in turn, that also helps build further conceptual understanding.

From the very beginning of the course, students have to reason from concepts (and prior knowledge) to answer questions. Questions that we pose prompt students to discover the concepts of the course as opposed to being "presented" with the concepts through reading the text or listening to an instructor's explanations. Following these initial discovery questions, we pose additional "synthesis" questions that provide students with the opportunity to adapt, extend and integrate their understanding of the concepts (in new settings) as this is an integral part of developing their ability to reason using those concepts. These synthesis questions may be embedded in an applied modeling investigation or they may be used to discover new concepts. In mathematics, as in any subject, a student is truly learning when they are asked to carefully examine their current understanding and then use it to develop new understanding.
An Example - The First Day of Class: Our precalculus students walking into their first day of class are surprised. The very first class activity they are faced with is the following:

**Problem 1:**

**Problem:** During criminal investigations, footprints are often used to attain a description of the suspect. In particular, a person’s footprint can be used to predict his/her height.

**Develop a Process:** Working together in small groups, design a process to predict, from a footprint, the height of the person who made that footprint. Discuss completely as a group and make sure that you all agree on the process. Then explain your process in detail below.

**Make a Prediction:** Using the process that you have developed, predict the height of the people who made the following footprints. Footprints (2-dimensional outlines of footprints – no depth) will be passed around the classroom.

The instructor hands out the activity and tells the students that there are measuring tapes and graph paper at the front of the room in case they need them for any reason during the activity. At first students are reluctant to do anything with the activity. They each read it individually and look nervously at the person on either side of them. Finally someone caves and discussion begins. However, after an initial few minutes of talking, some groups fall silent again. When the instructor approaches one such group and inquires what they are thinking about, they shrug and say they are not thinking because they do not know how to do the problem. The instructor encourages the group to try thinking about it again and asks if they knew that the footprint was bigger than their own footprint would that help them? What if it was smaller? This starts the group talking again and the instructor moves on. The classroom starts to look a little chaotic as a number of students have started to measure their own feet. Other students see them doing this and decide to give it a try themselves. One student is stretching her arms out as far as they can go on either side of her, and another student is holding her shoe up to her arm. When most groups seem to have come up with a process, the class comes together to discuss their different approaches. They debate how reasonable they think each approach is, compare and contrast the approaches, and in the end ensure that each group has an approach that they believe should give a good prediction. The following are two examples of approaches that students have come up with in our classrooms:

<table>
<thead>
<tr>
<th>Approach 1</th>
<th>Approach 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model:</strong> Height = 6.5 x footprint length</td>
<td><strong>Model:</strong> Height = 5.88 x footprint length</td>
</tr>
<tr>
<td>Student Explanation: The group explained: &quot;Your wingspan is the same as your height.&quot; Then based upon approximate measurements from their own bodies, they concluded that: &quot;Your foot is the same length as the distance from your wrist to your elbow which is roughly the same as the distance from your elbow to you shoulder. The distance between your shoulders is approximately one and a half times the length of your foot and your hands are each approximately half the length of your foot. Therefore</td>
<td>Student Explanation: &quot;There were four people in our group - Tijona, John, Rachel and Justin. For each person we measured their footprint length and height. These were the measurements that we got:</td>
</tr>
<tr>
<td>Tijona: Footprint Length = 10.2in, Height = 64in, Value = 6.27</td>
<td>Tijona: Footprint Length = 10.2in, Height = 64in, Value = 6.27</td>
</tr>
<tr>
<td>John: Footprint Length = 13 in, Height = 75in, Value = 5.77</td>
<td>John: Footprint Length = 13 in, Height = 75in, Value = 5.77</td>
</tr>
<tr>
<td>Rachel: Footprint Length = 11.5in, Height = 68in,</td>
<td>Rachel: Footprint Length = 11.5in, Height = 68in,</td>
</tr>
</tbody>
</table>
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Notice: Approach 2 may seem more mathematical (because they are doing some numerical calculations) but they are assuming proportionality (that we will be able to multiply the length of the footprint by a constant and get the height) without any justification. Whereas Approach 1 comes up with their proportionality statement from a physical process which inherently provides a justification for that statement and therefore makes it more mathematically correct. Justifying/recognizing specific types of relationships (e.g., proportional, linear, exponential, etc.) is a continual conceptual focus throughout this course. Students soon become far less willing to assume a relationship between two changing quantities without justification.

The students now try out their prediction processes on the three sample footprints, making their predictions for the heights based on measuring each of the footprints provided and applying their group's approach. The instructor creates a large table on the board and each group writes up their predictions for each of the sample footprints. The instructor then finally reveals the actual height of the three people whose footprints were used. Some of the groups' predicted heights are quite good for one of the footprints but none are close for the other two. The class then discusses why this might be the case and after some discussion suggestions arise around the data samples that each group collected - the size of the sample, whether the sample was chosen from an appropriate population of individuals, and of course the reasonableness of their prediction processes in the first place. The instructor at this point shares that the three footprints belonged to a child, a 28 year old woman and the other to a 20 year old man with large feet for his height. Given the discussion about the sample population used, students can now see some reasons for the large error in two of their predictions. The class concludes with a review of all of the steps they went through to try to answer this question (and with learning some associated modeling language - e.g., "data", "sample", and "model"): 1) thinking about the question, 2) collecting data to help them answer the question, 3) analyzing that data in a variety of ways, 4) coming up with a model for the situation, 5) using that model to make a prediction and finally 6) evaluating the accuracy of their model and the limitations of their model. The students are sent away to write up their complete solution and hand it in during the next class.

As this is the first day of class, the activity is not intended to immediately jump into the concepts of the course but instead to ask the students a "good question" that engages them from the very start of the course in the types of reasoning they'll be doing on a daily basis. This question is open, unfamiliar and unguided so as to force students to reach for their own understanding of the situation, and to use that understanding to make sense of the question and to develop an approach to answering the question. This question is set in an applied and familiar context as many of our questions are. This allows students to feel comfortable conjecturing and experimenting based upon their own prior knowledge about the situation being discussed in the question. They feel that they at least understand the situation and can picture it even if they are not sure what to do with the question yet. The applied context also allows
them to reach for data to help them answer the question which is a key idea that we will return to time
and time again in this course. This question highlights for students that they themselves have a
responsibility for evaluating how good their answer is: Have they taken into consideration the right
population and the right assumptions? Did they have enough data, and did they have evidence to
assume the type of relationship between the variables that they used? Most importantly though, this
question allows for multiple approaches and multiple answers - all of which could be correct. As a result
students must articulate and explain their reasoning to the class so that everyone can learn from their
approach. This articulation, in turn, is an extremely important step in fully developing their own
understanding. As instructors, we take the time to highlight a correct aspect of every group's approach
even if they have not developed their idea fully and praise the group for this work. This can be very
powerful for students who have always considered themselves terrible at math. The startling realization
that many students come to on the first day is that there is more than one right answer and that theirs
was one of the right answers!

For this day to set the stage for developing conceptual understanding, we must not only ask a good
question but also introduce our students to the collaborative classroom environment that we will use
for the rest of the semester. On the first day, students learn that when they read a question they will not
be expected to "know the method" but instead to identify what they know and what confuses them
about the question. They realize that this is just the first step towards answering the question. After this
initial examination, they will have to be prepared to discuss their reasoning with a few classmates and
resolve some of their confusion during this small group discussion. They will also be expected to share
their small group discussion with the whole class. They learn that the role of the instructor during whole
class discussion will not be to tell students that they are right or wrong but instead to help the class to
identify a number of key ideas from the small group work that the whole class should pursue. The
students know that following class discussion, they will often return to work further on the question in
their small groups or individually where they will be expected to try out some of the ideas that they or
their classmates had. Finally, they learn that they must rely on their own thinking because the instructor
is not going to suddenly show them "the method" for the question they are working on.

Following this first day's class, we spend a number of weeks where the students discover the concepts
needed to understand and model relationships between two varying quantities. The following example
(typically done during the first week of classes) illustrates how we have our students discover a concept
by working from prior knowledge as opposed to first explaining/defining the concept to the students.

<table>
<thead>
<tr>
<th>Example 2:</th>
</tr>
</thead>
</table>
| **Concept (not stated to class until after investigation below):** A linear relationship has a constant
  amount change per year and an exponential relationship has a constant percent change per year. |
| **Prior knowledge:** Students know what a linear relationship and exponential relationship look like
  graphically from their experiences in high school, but they have not yet learned any characterizations of
  these relationships in this course. |
| **Activity:**                                   |
| 1. Create possible data and a continuous graph to illustrate a situation in which the population of a
town is increasing at a constant rate. |
2. Create possible data and a continuous graph to illustrate a situation in which the population of a town is increasing faster and faster.

3. Create possible data and a continuous graph to illustrate a situation in which the population of a town is increasing by the same percent each year.

Discussion:
After working on these questions, students notice that the graph they have created for Question 1 is a line which leads them to believe that this type of relationship between population and time is a linear relationship. The instructor asks how we could have recognized this from the data instead of from the graph. Students discuss this further and conclude that the key point was that the population was increasing by the same amount each year which they could check from their data. Further discussion is needed to decide what to do if you didn't have data for every year - how would you still know it was increasing by the same amount each year.

Question 2 and 3 are more confusing for the students - both of these graphs look like exponential graphs to them therefore they conclude that both could be exponential relationships. The instructor guides the students by telling them that their graph in Question 3 is exponential but that the graph in Question 2 is not exponential in every group. The students are asked to revisit the two sets of data and to try to figure out what is different between them. Students focus in on the fact that although both populations are growing faster and faster only the Question 3 population has the added restriction that it must grow by the same percent each year. They decide that this must be the defining characterization needed for an exponential relationship.

Follow-up: In a later activity the students will come back to the discoveries made here (numerical and graphical characteristics) to build symbolic equations for both linear and exponential relationships through a similar questioning process.

After the work during these weeks, students understand linear relationships graphically (the graph of the data is a straight line), numerically (the rate of change between any two points is the same), in narrative terms (descriptions of linear relationships) and through symbolic equations ($y = mx + b$). The students are then ready for the following question - similar to the question assigned on the first day:

Problem 2:

Question: You want to surprise your Dad by buying him a new shirt for Father’s Day, but you do not know his neck size. You do, however, have his watch (since you borrowed it the other day and forgot to give it back) – from which you could measure his wrist size.

Develop a Process: Working together in small groups, design a process to predict the neck size of a person from the size of their wrist. Discuss completely as a group and make sure that you all agree on the process. Then explain your process in detail below.

Make a Prediction: Using the process that you have developed, predict your Dad’s neck size if, using his watch, you discovered that his wrist size was 21 cm.
What is interesting in this activity is how the students' approach has changed from the first day's footprint activity. They are willing to quickly jump to collecting data, but many groups immediately dismiss the possibility that this relationship is linear as the slope between pairs of points is not the same (and can sometimes be vastly different). Remember that on the first day, many were slow to think about collecting data but as soon as they had data immediately reached for a type of linear relationship without any justification for doing so. These students have developed an understanding of the concept of linear functions (if the rate of change is not the same between every pair of points the relationship is not linear) but still need to further develop the ways in which they can adapt and expand on the use of this concept so as to answer new questions. The groups who graph the data see that (despite the numerical evidence to the contrary) the data falls roughly along a straight line and share these findings with the class. This allows everyone to re-evaluate what we mean by a linear relationship and whether we need to expand on the use of that concept to model the relationship we are seeing between the wrist and neck measurements. By the end of this activity, students understand that we can describe this relationship as being "somewhat linear" or "well-represented by a linear function" and that in these situations we can use our understanding of linear functions to help us answer questions about this type of relationship.

In the above activities, we see how the students are building their own understanding of a concept (numerical, graphical, narrative and symbolic characterizations of linear relationships) reasoning from prior knowledge, and then expanding on that understanding to investigate a new concept (modeling approximately linear relationships from data). Most concepts in our precalculus course are encountered by our students through this same development process, simultaneously developing in our students both a deep understanding of the meaning of each concept and the ability and confidence to engage in the challenging process of independently reasoning directly from their own understanding to answer questions.

We help our precalculus students build **conceptual understanding** by actively engaging them on a daily basis in the process of having to reason from concepts (and prior knowledge) to answer questions and solve problems, both within applications and within purely mathematical contexts. How?

1. **Questioning** that forces students to actively engage in reasoning from concepts:
   - Open, unfamiliar, unguided questions.
   - Questions that require students to articulate and explain their reasoning.
   - Students work on questions (individually or in small groups) prior to establishing consensus as a class - both when developing an initial understanding of the concept and when expanding/applying that understanding in synthesis.

2. **Learning Environment** that supports and promotes the development of reasoning from concepts:
   - Collaboration: In pairs, small groups, together as a class, and together with the instructor

Expectations and Trust: Students know what is expected of them, they know nobody (including the instructor) is going to just give them the answer or provide example solutions, and they trust that the class will eventually come together with the Instructor to clarify points of confusion.
Assessing Conceptual Understanding

As instructors, we focus a huge amount of energy on providing the course activities and classroom environment that we believe will help our students develop conceptual understanding. However, for us to truly know the effectiveness of this work we must be equally invested in the assessment of this conceptual understanding. We quickly discover that it is not easy to assess conceptual understanding, as students could correctly answer many questions intended to test a concept by using a method that they have developed or memorized while working on a similar question in previous course work. Therefore, to truly assess conceptual understanding we must examine a concept in a setting that is novel enough to require that the student is indeed reasoning from an understanding of the concepts in order to correctly answer the questions.

This is easier said than done. Many students enter our class with the opinion that being given a problem which they haven't seen before on a homework or an exam is not fair. In a traditional classroom, they might even be justified as they have probably not had an opportunity to develop the ability to reason from a concept to solve an unfamiliar problem. In order to prepare students for these novel problems, we must first carefully build both their confidence to attack problems that are not of a "type" they recognize and their ability to use their understanding of concepts to answer these questions. We do this by engaging students in this type of reasoning in every class activity and in every homework set. We also discuss regularly with our students how important it will be in their future careers to be able to approach an unfamiliar problem and form a strategy for answering it. After this work in class and on homework, we can ask a question on an exam that involves a novel context or use of a concept without the reasoning experience itself being novel or unexpected.

Students are now prepared to see novel problems on homework and exams which leaves us with the task of writing these problems! After identifying the concepts we want to test over, we attempt to write a novel question for the exam. For instance, consider the concept of characterizing linear and exponential growth/decay which the following question from a course activity helped to develop:

**Problem 3**

*Escherichia coli*

The data in the table below was taken from a patient who presented themselves to the hospital emergency room with an initial count of 2000 *E. coli* cells. A particular antibiotic was administered every three hours. Listed below are the number of *E. coli* cells that remain in the patient at the beginning of each 3 hour period (t hours after the first dosage of antibiotic).

<table>
<thead>
<tr>
<th>Time (t) in hours</th>
<th>Population of Bacteria Remaining</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2000</td>
</tr>
<tr>
<td>3</td>
<td>1372</td>
</tr>
<tr>
<td>6</td>
<td>941</td>
</tr>
<tr>
<td>9</td>
<td>646</td>
</tr>
<tr>
<td>12</td>
<td>443</td>
</tr>
<tr>
<td>15</td>
<td>303</td>
</tr>
<tr>
<td>18</td>
<td>208</td>
</tr>
<tr>
<td>21</td>
<td>143</td>
</tr>
<tr>
<td>24</td>
<td>98</td>
</tr>
</tbody>
</table>

**Question:** Determine how many bacteria remain in this patient after 48 hours of treatment.

This class activity question helps build students' conceptual understanding by requiring students to use their understanding of linear and exponential growth/decay (reason from the concepts) to decide if this data exhibits characteristics of either type of growth/decay, and it is the first time they've had to make
the connection between the meaning of this mathematical characteristic and how to look for this characteristic in a set of data. In addition they have to articulate why the approach that they use allows them to draw these conclusions. When faced with writing an exam question that tests this same concept one option is to do so in the context of a new applied setting. The following is an example of such a question:

**Problem 4:**
Megan graduated from college in 1980 and starts working that year. Below you will see a table of data showing her annual income, $I$, at various stages in her career $t$ years after 1980. Note that the data in the table assigns $t=0$ to the year 1980.

<table>
<thead>
<tr>
<th>Time, $t$ years since 1980</th>
<th>Annual Income, $I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>22675</td>
</tr>
<tr>
<td>5</td>
<td>26448</td>
</tr>
<tr>
<td>7</td>
<td>30849</td>
</tr>
<tr>
<td>9</td>
<td>35982</td>
</tr>
<tr>
<td>11</td>
<td>41970</td>
</tr>
<tr>
<td>13</td>
<td>48954</td>
</tr>
</tbody>
</table>

**Question:** Quantitatively evaluate whether this data would be well represented by a linear function, an exponential function or neither. Show your calculations, state your conclusions and explain your reasoning here.

Although this is a different applied setting from the one originally encountered in class and the data is increasing instead of decreasing, it is possible for a student to answer this question by simply repeating the same sequence of steps (the method) that they developed and performed on Problem 3 on the course activity without ever considering why they chose that sequence of steps the first time. This means that the student is no longer using (and thus, demonstrating) their ability to reason from an understanding of the concepts involved. Most likely, they are instead recognizing that this problem is of a very similar format to the problem on the course activity and then adapting their method from that problem to answer this one. As a result, a problem that might have assessed conceptual understanding very well in another precalculus classroom, becomes more of a method problem in our classroom because of our students' previous experience with that concept in the context of our course. However, one small change in the data provided in this problem - for example, giving incomes for years since 1980 that are not in consistent increments (e.g. 3, 8, 10, 14 instead of 3, 5, 7, 9, 11 etc) renders the method developed in the course activity no longer applicable in this problem. The student would then be forced to return to their understanding of the mathematical concept in order to answer the question. Notice also that neither the applied setting nor any alternate representations (only numerical is present) play a significant role in either the original problem or our adapted problem. We could improve this problem further by requiring that the student either interpret the meaning of the mathematical characteristic in the context of the applied setting or make connections between different representations (numerical,
graphical, symbolic or narrative) of the mathematical characteristic. Notice the problem below (which assesses the same concept) requires students to do both.

Problem 5: Megan graduated from college in 1980 and starts working that year. Below you will see a table of data showing her annual income, $I$, at various stages in her career $t$ years after 1980. Note that the data in the table assigns $t=0$ to the year 1980.

```
<table>
<thead>
<tr>
<th>Time, $t$ years since 1980</th>
<th>Annual Income, $I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>22675</td>
</tr>
<tr>
<td>8</td>
<td>33317</td>
</tr>
<tr>
<td>10</td>
<td>38861</td>
</tr>
<tr>
<td>14</td>
<td>52869</td>
</tr>
<tr>
<td>17</td>
<td>66600</td>
</tr>
<tr>
<td>27</td>
<td>143785</td>
</tr>
</tbody>
</table>
```

Choose all of the following descriptions below that could describe the relationship between years since 1980 and Megan's annual income:

1. Megan's income increases by $10,642 dollars every five years.
2. Megan's income increases by 9.39% every year.
3. $I(t)=10642t$
4. 
5. Megan's income is increasing by more and more each year.
6. The rate of change of Megan's income over time is decreasing.
7. $I(t)=14289(1.08)^t$
8. $I(t)=14289(t)^{(.08)}$
9. 
10. Megan's income increases by 8% every year.
11. Megan's income increases by $7718.5$ every year.
12. $I(t)=22675(.92)^t$
13. 
14. Megan's income increases at a constant rate.

---

2 In "Understanding by Design", Wiggins and McTighe "have developed a multifaceted view of what makes up a mature understanding, a six-sided view of the concept." The six facets that the mention are "can explain, can interpret, can apply, have perspective, can empathize and have self-knowledge". In the questions that we describe as assessing conceptual understanding in mathematics well, students are demonstrating the first three of these facets.
While considering how well the exam questions that we had written assessed our students' conceptual understanding, we realized that we were asking ourselves a series of questions about each problem. We decided to try to formalize this questioning process so that we would then have a tool for evaluating the extent to which conceptual understanding is being assessed by a given problem in our course. That tool could then be used to weight the scoring of exam problems, giving us scores for our students (separate from their exam scores) that indicate more specifically their level of conceptual understanding on that exam. In addition, this tool also gives us the ability to track our students' proficiency at specific types of mathematical understanding.

To begin with, we recognized that in every math course there are computational and/or algebraic skills needed for the work in that course. We want our students to become proficient with these skills, so we put some problems on exams that assess these skills. These skills problems, however, do not require any conceptual understanding so our weighting system should assign them a weight of zero. We then began to realize that on some of the problems that we had specifically written to test students' conceptual understanding, students were in fact more likely answering the problem by adapting methods that they had developed previously in class work (as seen in Problem 4 above). Being able to adapt a method in this way demonstrates some conceptual understanding but not as much as when a student must reason directly from the meaning of a concept. Our weighting system should assign these types of problems a weight of one. If no previously developed method can be applied to solve the problem, we assume that students must be reasoning from concepts in order to answer that question. Our weighting system should assign a weight of at least two to these types of problems. If in addition students were required to interpret the meaning of a mathematical characteristic in a novel applied setting, and/or the students were required to make connections between different representations (numerical, graphical, symbolic or narrative) of a mathematical characteristic then students are demonstrating an even deeper understanding of concepts when successfully answering that question. To reflect this, these types of problems should receive a weight of three. This Conceptual Understanding Weighting System (CUWS) weights each problem (weights ranging from 0 to 3) according to the extent to which the problem assesses students' conceptual understanding - depending upon both the characteristics of the problem itself and how a student has previously encountered (in this course) the concept/s being tested by the problem.

### Conceptual Understanding Weighting System (CUWS)

**Check 1 (Skills):** Does this problem involve only computational/algebraic skills or memorized facts with no understanding of the concept required? Yes - assign a weight of 0 to the problem and skip the remaining checks; No - continue to the next check.

**Check 2 (Method):** Could this problem be answered completely using a method that the student or teacher might have developed in prior course work? Yes - assign a weight of 1 to the problem and skip the remaining checks; No - continue to the next check.

**Check 3 (Conceptual Reasoning Characteristics):** Does the problem involve either of the following? No - assign a weight of 2 to the problem; Yes - assign a weight of 3 to the problem.

- interpreting the meaning of a mathematical characteristic in a novel applied setting
- making connections between different representations (numerical, graphical, symbolic or narrative) of a mathematical characteristic
Examples Applying CUWS:

*Problem 4 above* - Weight = 1 as this is a question that could have been answered completely with a method developed in prior course work

*Problem 5 above* - Weight = 3 as this is a question that could not have been answered with a method developed during the course and it satisfied both of the bullets under Check 3. It asked students to make connections between different representations of mathematical characteristics - graphical to numerical, symbolic to numerical etc., and it asked students to interpret the meaning of exponential change in a novel applied setting (annual income over time).

Results of Assessment

In order to evaluate how successful we had been in helping our precalculus students build conceptual understanding, we assessed our students' conceptual understanding in two main ways: 1) semester-long, course embedded assessment (using our course exams), and 2) pre-post assessment of standard precalculus concepts (using the Precalculus Concept Assessment\(^3\) (PCA)). Student scores (on course exams and the PCA) reflected different types of mathematical understanding. We applied the Conceptual Understanding Weighting System described above to our course exams and to the PCA results in order to organize our assessment data and extract information about our students' proficiency in specific categories of mathematical understanding. For each assessment, we calculated both conceptual scores (weighted) for each student and levels of class proficiency in specific categories of mathematical understanding.

Below are sample (combined) results for the authors' Precalculus Fall 2009 sections:

**Exam 1 Results**

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>CUWS Weight</th>
<th>Average Percent Score on Each Question (58 students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1</td>
<td>83.79</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1</td>
<td>84.14</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>1</td>
<td>79.14</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>1</td>
<td>76.01</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>1</td>
<td>67.67</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>1</td>
<td>81.84</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>3</td>
<td>72.52</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>3</td>
<td>66.72</td>
</tr>
</tbody>
</table>

For each student we can consider both a raw score, a weighted score and proficiency levels within the

\(^3\) The Precalculus Concept Assessment was developed by Marilyn Carlson at Arizona State University with funding from the National Science Foundation and Arizona State University.
Building Conceptual Understanding in Precalculus

different weighting categories. For example consider a student who got the following scores (in percent) on each question of Exam 1: 100, 100, 90, 91.67, 62.5, 100, 100, 40. Their raw score on the exam is 88% (= (1.00*10+1.00*10+0.90*20+0.9167*12+0.625*8+1.00*15+1.00*15+0.40*10)/100) whereas their weighted score using CUWS is 84% (= (1.00*10+1.00*10+0.90*20+0.9167*12+0.625*8+1.00*15+1.00*(15*3)+0.40*(10*3))/150). This tells us that this student has not done as well on the questions assigned a weight of 3 as they did on the questions assigned a weight of 1. This is also demonstrated by calculating a proficiency level for this student within each weighting category. For questions assigned a weight of 1 the student scored 92% (=1.00*10+1.00*10+0.90*20+0.9167*12+0.625*8+1.00*15+1.00*(15*3)+0.40*(10*3))/75 whereas for questions assigned a weight of 3 the student scored 76% (=1.00*15+0.40*10)/25).

The mean student score on Exam 1 without any weighting was 76.98%. The mean weighted score when using the CUWS weightings provided in the table above was 74.72%. After applying the CUWS to Exam 1 it is evident that it mainly assessed a student’s ability to adapt a previously learned method (reflected in the weighting above where six questions were assigned a weight of 1 whereas only two questions were assigned a weight of 3). Although adapting a previously learned method is something we want our students to be able to do, it is an intellectual task requiring only minimal conceptual understanding as discussed above. We would prefer that our exams instead have types of questions from each of the weighting categories mentioned above - skills questions, method questions, conceptual questions and questions that are assessing deep conceptual understanding. However the difference between the weighted score (74.72%) and the raw mean (76.98%) still demonstrates that students are finding the questions weighted 3 more difficult. This is also reflected when we consider the overall mean proficiency score within each weighting category: mean on questions of weight 1 = 79.24%, mean on questions of weight 3 = 70.21%. Although we consider 70% to be a very good mean score on deep conceptual questions over a diverse group of students we hesitate to draw any conclusions from the results of this exam due to the fact that there are only 25% of the questions (and points) assigned a weight of 3.

Exam 2 Results

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>CUWS Weight</th>
<th>Average Percent Score on Each Question (54 students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1</td>
<td>84.44</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>3</td>
<td>82.92</td>
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<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>82.96</td>
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</tr>
<tr>
<td>6</td>
<td>10</td>
<td>1</td>
<td>83.70</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>3</td>
<td>60.88</td>
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<tr>
<td>8</td>
<td>15</td>
<td>3</td>
<td>50</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>1</td>
<td>73.33</td>
</tr>
</tbody>
</table>

The mean student score on Exam 2 without any weighting was 65.88%. The mean weighted score when
Building Conceptual Understanding in Precalculus

using the CUWS weightings provided in the table above was 60.87%. This points to the fact that our students are doing better on the questions weighted 1 (adapting a method) than the questions weighted 3 (requiring deep conceptual understanding). This is again supported when we consider the proficiency level in each of these weighting categories. 65% of students scored 70% or higher on the questions weighted 1 whereas only 33% of students scored 70% or higher on questions weighted 3. This reaffirms our belief that students are more proficient at adapting a method that they have already used than answering a truly novel question and shows that on this exam 33% of our students demonstrated deep conceptual understanding.

Precalculus Concept Assessment (PCA) Results

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>CUWS Weight</th>
<th>Number of Students with this Question Correct (Out of 50 students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>1</td>
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<tr>
<td>10</td>
<td>4</td>
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</tr>
<tr>
<td>11</td>
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<td>1</td>
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<tr>
<td>15</td>
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</tr>
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<td>3</td>
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<tr>
<td>21</td>
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<td>1</td>
<td>3</td>
</tr>
<tr>
<td>22</td>
<td>4</td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

The PCA is an assessment that the students do not study for, that does not factor into their course grade, and that poses a high level of difficulty for most of our students. With this lack of preparation or incentive, it was not surprising that most of our students did not score very well on this assessment. However, one interesting (and unexpected) result did emerge when proficiency levels within each weighting category were compared: On questions assigned a weighting of 1 the mean student score was 27.5%; on questions assigned a weighting of 2 the mean student score was 38%; and on questions assigned a weighting of 3 the mean student score was again 27.5%. This is very different from the results seen in course exams (Exam 1 and Exam 2) where students were performing better by far on the questions weighted 1. It makes sense that on a test that students have not studied for, they are not
drawing on previously developed methods to answer the questions but instead are more likely reasoning from concepts. It is also interesting that here (where for the first time we see the distinction between questions weighted 2 and questions weighted 3) students are doing significantly better on the questions weighted 2 than on the questions weighted 3.

Final Thoughts

Over the course of writing this chapter we have discovered a great deal about both teaching and student learning in our classrooms. We have come upon these discoveries in a variety of ways - some from the act of articulating what exactly it is we do in course activities to allow students to develop their understanding of a concept, some from examining exactly how we assess (or have failed to assess in some cases) that conceptual understanding, and still more from analyzing the data that we collected over the course of Fall 2009.

- We are happy with our current approach to developing conceptual understanding as a whole. However based on the analysis of exam questions and the PCA, we have already planned improvements to many of our course activities. In addition, we plan to create some new course activities to address concepts that we had not previously discussed in this course. We will also map each exam and PCA question to the course activity/activities that developed the concepts being assessed in that question. With this mapping we can then use student results on the exams and the PCA more effectively to identify the course activities that are not building conceptual understanding to the extent that we would like and revise those course activities accordingly.
- Our assessment of conceptual understanding is still a work in progress. The process of writing the chapter has opened our eyes to the limitations of our current exams. Using our new CUWS to help us evaluate the questions as we write them should enable us to improve our assessment of conceptual understanding not just in Precalculus but also in our other math courses.
- Our development of the CUWS itself is only in the preliminary stages. We have developed a student Survey of Mathematical Thinking (SoMT) that is administered to students after the exam. This asks students to describe how they approached some of the questions on the exam. We can in turn use this to compare to our weighting with the CUWS to see if students are approaching questions in the way that we thought. We have administered this survey once and are analyzing the data currently.
- All of our work to develop and assess conceptual understanding is part of a larger project working on these same questions in many courses that we teach. We have developed a student centered, active-learning curriculum for Calculus I and II using the same ideas discussed in this chapter and plan to use the CUWS to help in assessing conceptual understanding in these courses as well.4
- In the course of our research, we are more effectively engaging our students as partners in the learning process. As we ourselves have come to better understand the subtleties of building and assessing conceptual understanding, we have been able to communicate more clearly to our students why we do the things we do in our course and what their role is in the learning experience.

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4 All of our course activities for Precalculus, Calculus I and Calculus II are available online. Please contact the authors for further details.
Reflection

As with much SOTL work, we did not begin this work with the goal of developing an SOTL project. For many years in our department there has been a firm belief that actively engaging students during class time improves conceptual understanding. Recently, however, we began to notice that although students were active during class time they were sometimes not developing the level of conceptual understanding that we expected. We reflected more carefully on what exactly students were actively engaged in during class and we decided that in many cases, they were still not being asked to reason from concepts to develop their own understanding. We realized that we were guiding the students so much in the course activities that we were still the ones engaging in this reasoning process and the students were simply following our plan while missing the big picture. We decided to revise each of our activities to try to incorporate a more open approach that would allow students to develop their own understanding of a concept. We were very happy with the results and how these new activities worked in our classroom but at the same time we were unable to be certain that students were, indeed, developing deep conceptual understanding. This was probably the first time that we began to think of this project as an SOTL project. We wanted to investigate whether the understanding that we believed was being developed truly was. We developed and implemented a plan to collect data to investigate our claim using an external concept exam as well as course embedded assessment. When analyzing this data we realized another difficulty: How could we identify which questions were assessing a student’s ability to reason from concepts as opposed to assessing our students proficiency with skills or ability to adapt a mathematical method? We studied our exam questions carefully and found that many of the questions we had written to assess conceptual understanding were not in fact as good as we had hoped. This was disappointing at first but it led us to the interesting problem of how to identify and write problems which require students to reason from concepts. In doing this we developed the Conceptual Understanding Weighting System which not only helps us to focus in on the evidence of conceptual understanding in our current data but also we believe will help us to write better exam problems in the future. The most surprising consequence of our work on this project was the direct impact approaching our questions from an SOTL perspective had on our classroom experience. Developing our research questions, designing our investigation plan and then collecting, analyzing and drawing conclusions from data directly impacted how we now write activities, exams and homework thus having a direct impact on our students' learning experience.